PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Path-Dependent Options and Transaction Costs

J. N. Dewynne, A. E. Whalley and P. Wilmott

Phil. Trans. R. Soc. Lond. A 1994 **347**, 517-529
doi: 10.1098/rsta.1994.0061

Path-dependent options and transaction costs

By J. N. Dewynne[1], A. E. Whalley[2] and P. Wilmott[2, 3]

[1] *Department of Mathematics, University of Southampton, Highfield, Southampton SO9 5NH, U.K.*

[2] *Oxford University Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, U.K.*

[3] *Department of Mathematics, Imperial College of Science, Technology and Medicine, London SW7 2AZ, U.K.*

We describe research in the subjects of exotic option pricing and option pricing when trade in the underlying incurs transaction costs. These two subjects are then formally brought together to model, in terms of differential equations, problems in pricing exotic options with transaction costs. Results are presented in several cases.

1. Introduction

There has recently been a great deal of interest both in the academic literature and among practitioners in the two subjects of option pricing in the presence of transaction costs and valuing exotic options. As we shall shortly see, valuing options when there are costs associated with trading the underlying leads to some interesting mathematical problems with important consequences. With the current popularity of exotic options it is clearly important to be able to price such products in the same framework. As exotic options become more frequently traded so profit margins will be squeezed and the trader with the most accurate models will make the most profit. Part of this accurate modelling is knowing the effect of transaction costs on an option's price.

In §2 of this paper we describe the modelling of exotic option prices as the solution of partial differential equations. We set up a very general framework into which we can incorporate many path-dependent options including Asians and lookbacks. We derive the relevant partial differential equations along with boundary, final and jump conditions and constraints.

In §3 we describe in detail the generalized Leland model for the effect of transaction costs on vanilla options. We then review other partial differential equation models for other trading strategies. We assume throughout a fairly general transaction cost structure.

In §4 we put together the results of §§2 and 3 in a partial differential equation model for pricing exotic options with transaction costs. We briefly discuss the possibility of similarity solution and the effects of discrete sampling of the path-dependent quantities.

Some results of numerical simulations for discretely sampled lookback strike puts are presented in §5.

2. Pricing exotic options

(a) *Some common exotics*

The value of an option depends on its pay-off at exercise or expiry. In the case of vanilla calls this pay-off is simply $\max(S-E,0)$, where S is the price of the

Phil. Trans. R. Soc. Lond. A (1994) **347**, 517–529
Printed in Great Britain

underlying asset and E is the exercise price, a given constant. From this most basic of option contracts it is a small step to the binary option. Such an option has a more general pay-off, for example $V(S,T)=\mathscr{H}(S-E)$, where V is the price of the option, T is the expiry date and $\mathscr{H}(\cdot)$ is the Heaviside function. (This can be interpreted as a straight 'bet' on whether the asset price will be above or below E at expiry.) The next stage in increasing complexity is the barrier option. A simple example of a barrier option is a 'down-and-out'. This is similar to vanilla option but with the extra specification in the contract that should S ever fall to a prescribed level X, say, before expiry, then the option becomes worthless.

American options (which permit early exercise, as opposed to European options which do not) and barrier options are 'path-dependent': the value depends on the realized asset price path. However, this dependency is rather simple and it is still possible to write the option price as a function of S and t only, $V(S,t)$. Examples of option contracts that are path-dependent in a non-trivial manner are Asian options and lookback options. The former option has a pay-off depending on an average of the realized asset price random walk and the latter on the realized maximum (or minimum).

In saying that the Asian option pay-off depends on the average of the asset price we must be very precise in our definition of 'average'. Two obvious definitions exist for an average (and there are many more less obvious ones): the arithmetic average and the geometric average. These two types of average are distinguished by whether the mean is taken of the asset prices or the logarithm of the asset prices. Finally, the average may be continuous or discrete, depending on whether all asset prices are used in the mean or only those at prescribed 'sampling dates'. The lookback option may similarly depend on either the continuously sampled or the discretely sampled maximum (or minimum).

(b) *The general framework*

The options above can all be put into the basic Black–Scholes framework (Black & Scholes 1973) with very little effort.

Let us suppose that the option contract specifies a pay-off which is a function of S and an integral over the underlying's realized history of the form

$$I=\int_0^t f(S(\tau),\tau)\,\mathrm{d}\tau.$$

Thus the pay-off at exercise has the form $\Lambda(S,I,t)$. For example, when

$$f=S,$$

this gives a dependence on the continuously measured arithmetic average. When

$$f=\log S,$$

we get the continuously sampled geometric average. When

$$f=S\sum_{i=1}^{N}\delta(t-t_i),$$

we have the discretely sampled arithmetic average with N sampling dates t_i. When

$$f=\log S\sum_{i=1}^{N}\delta(t-t_i),$$

we have the discretely sampled geometric average and, finally, when

$$f = S^n,$$

(multiplied by delta functions if necessary) and

$$J = \left(\int_0^t f(\tau)\,\mathrm{d}\tau \right)^{1/n},$$

we get, in the limit $n \to \infty$, the maximum of the asset price, or as $n \to -\infty$, the minimum. (Thus both Asian options and lookback options have pay-offs which depend on I or J, provided f is suitably defined.)

Following the derivation of the original Black–Scholes equation we assume that S follows the random walk given by the geometric brownian motion

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}X,$$

where $\mathrm{d}X$ is a normally distributed random process with $E(\mathrm{d}X) = 0, E(\mathrm{d}X^2) = \mathrm{d}t$. We set up a portfolio of one option, with value $V(S, I, t)$, and $-\Delta$ of the underlying asset. If Π is the value of this portfolio then

$$\Pi = V - \Delta S.$$

From Itô's lemma we have

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta\,\mathrm{d}S = (V_t + fV_I + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - \Delta D(S, I, t))\,\mathrm{d}t + (V_S - \Delta)\,\mathrm{d}S,$$

where D is the dividend on the underlying asset: an amount $D(S, I, t)\,\mathrm{d}t$ is paid on the asset from t to $t + \mathrm{d}t$. We choose $\Delta = V_S$ to eliminate the random terms. This leaves a deterministic portfolio which, in the absence of early exercise, must have a return equal to that from a risk-free deposit. Thus

$$\mathscr{L}(V) = V_t + fV_I + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (rS - D)\,V_S - rV = 0. \tag{1}$$

(c) *Constraints*

We denote the general early pay-off for a path-dependent American option by $\Lambda(S, I, t)$. The absence of arbitrage opportunities implies that we must always have

$$V(S, I, t) \geqslant \Lambda(S, I, t).$$

If early exercise is possible, then (1) is only valid if early exercise is not desirable, that is when

$$V(S, I, t) > \Lambda(S, I, t).$$

If early exercise is desirable, this is so because it is more profitable to turn the option into its cash equivalent and put the funds in a bank; the implication is that $\mathscr{L}(V) < 0$ when early exercise is desirable. Thus,

$$\mathscr{L}(V) \leqslant 0;$$

the Black–Scholes differential operator can be seen to measure the difference between the riskless return on a hedged portfolio and the riskless return on a bank deposit; hence the equality when it is optimal to hold the option and the inequality when it is optimal to exercise it early. We may therefore cast the American option problem in the linear complementarity form

$$\left.\begin{aligned} V(S, I, t) \geqslant \Lambda(S, I, t), \quad \mathscr{L}(V) \leqslant 0, \\ (V(S, I, t) - \Lambda(S, I, t))\,\mathscr{L}(V) = 0. \end{aligned}\right\} \tag{2}$$

Uniqueness of the solution follows from continuity of V and V_S, prescription of the final value of V from the pay-off function $V(S,I,T)$ and boundary conditions that may apply as a result of barrier features. The linear complementarity form leads naturally to the formulation of the problem as a variational inequality, from which existence and uniqueness results may be easily deduced (see Wilmott *et al.* 1993).

(d) *Final conditions*

Equation (1) and the partial differential operator in (2) are backward parabolic and thus require final data to be specified at expiry, $t = T$. The appropriate final data is obviously the pay-off at expiry. Thus

$$V(S,I,T) = \Lambda(S,I,T).$$

The general form for Λ allows call, put and binary option varieties of exotics as well as of vanillas.

(e) *Boundary conditions*

For options with no barrier features, (2) must be solved in $0 \leqslant S < \infty$. In this case the transformation $S = S_0 e^x$, where S_0 is a typical value of S, maps $S = 0$ to $x = -\infty$, $S = \infty$ to $x = \infty$ and reduces the partial differential operator in (2) to a constant coefficient operator. This shows that we need not specify boundary conditions, other than a restriction on the growth as $x \to \pm\infty$. On the other hand, options with barrier features have boundaries at a finite value of S. In this case boundary conditions must be imposed. An example is a down-and-out barrier option, which we must solve on $X \leqslant S < \infty$, with $V = 0$ on $S = X$, the barrier.

(f) *Jump conditions*

When the average (for Asians) or the maximum and/or minimum (for lookbacks) is measured discretely, the governing partial differential equation contains delta functions in time. Except at sampling dates these vanish and the partial differential operator reduces to the usual Black–Scholes operator. Across sampling dates, t_i, the $\delta(t-t_i)\,V_I$ term can only be balanced by the V_t term, and hence across sampling dates we find that the option value locally satisfies the first order hyperbolic equation

$$V_t + f(S)\,\delta(t-t_i)\,V_I = 0. \tag{3}$$

This shows that the option value jumps across sampling dates and that a jump condition must be applied. These can be arrived at easily from equation (1) or alternatively by a simple financial argument. This argument is that the realized option price must be continuous across sampling dates. For example, for the discretely sampled geometric average option we have

$$V(S,I,t_i^-) = V(S,I+\log S,t_i^+).$$

This relates the value of the option before and after the sampling date t_i.

Similarly, if dividends are paid discretely, we can model this by a delta function in the dividend structure, $D(S,I,t) = \Sigma\,\delta(t-t_i)\,D_i(S,I)$. Across dividend dates the partial differential operator is approximated by

$$V_t - \delta(t-t_i)\,D(S,I)\,V_S = 0,$$

which again implies jump conditions. If the dividend structure is given, i.e. if $D(S,I)$ is given, then we can solve this equation across dividend dates to obtain the jump condition explicitly.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(*g*) *Further comments on pricing exotics*

More details about the partial differential equation approach to pricing exotic options can be found in Ingersoll (1987), Dewynne & Wilmott (1991, 1993) and Wilmott *et al.* (1993). There is one main advantage to this approach over the popular combination of equivalent martingale measure analysis and Monte Carlo simulation. This advantage is simply speed of computation. Once the problem has been formulated as a partial differential equation then we can apply quick numerical methods to calculate the price of an option as well as possibly to determine bounds on the error due to the discretization. Although the problems presented above are in three variables S, I and t, there are many important options for which there exist similarity reductions. This means that the numerical solution will generally be as quick as that for vanilla options; we shall see more of this later.

3. Transaction costs

(*a*) *Approaches to modelling the effects of transaction costs*

In the original Black–Scholes analysis and in that above for exotics, there are the assumptions of continuous rehedging and absence of transaction costs in trade in the underlying. Both of these assumptions are invalid in general and, depending on the liquidity of the market in question, may actually be very important. Several authors model option prices without these assumptions. In these circumstances we are faced with two distinct problems: determining the hedging strategy (when to rehedge) and valuing the option (given the hedging strategy). Rehedging will reduce, but not eliminate, risk, but at a cost. Whether we rehedge or not we must put a value to an inherently risky portfolio; we can no longer appeal to 'no-arbitrage'.

Two approaches have been taken in the academic literature: local in time and global in time. The former, for example Leland (1985), Boyle & Vorst (1992), Hoggard *et al.* (1993) and Whalley & Wilmott (1992), consider risk and return over a short interval of time. The latter, for example Hodges & Neuberger (1993) and Davis *et al.* (1993), adopt 'optimal strategies', in which risk and return are considered over the lifetime of the option. Both of these approaches have their advantages. The first group are easier to compute as they are only two-dimensional problems and, in particular, model market practice (Whalley & Wilmott 1992). The optimality of Hodges & Neuberger and Davis *et al.* is of obvious appeal, but leads to models that are usually impractically slow to compute. They also require input of the user's 'utility function', and very few practitioners adopt this approach. Recent unpublished research by Whalley and Wilmott suggests that simple asymptotic analysis of the global models, assuming that costs are in some sense small, yields nonlinear partial differential equations of the form encountered in the local models. Thus the two approaches may not be so different. For ease of exposition we shall describe the model of Leland and Hoggard *et al.* and then simply quote results from the other models.

(For an examination of a stochastic control problem in portfolio management with transaction costs see Morton & Pliska (1993), and for the asymptotic solution of that problem for small transaction costs see Atkinson & Wilmott (1993).)

(b) The generalized Leland model

We shall set up the model in a discrete time framework and assume that the asset price follows the random walk given by

$$\delta S = \mu S\,\delta t + \sigma S\phi\,\delta t^{\frac{1}{2}}, \tag{4}$$

where ϕ is a random number drawn from the standardized normal distribution. We are using $\delta\cdot$ to denote the small, but finite, change in a quantity. We can still write the option price as $V(S,t)$ and set up a portfolio as before with

$$\Pi = V - \Delta S,$$

with Δ to be chosen. We shall further assume that costs take the form

$$k_1 + k_2 N + k_3 NS,$$

where $N > 0$ is the number of shares traded to rehedge. Thus there are three components to the transaction costs: a fixed component (k_1), a cost proportional to the number traded ($k_2 N$) and a cost proportional to the value traded ($k_3 NS$).

Now consider the change in Π over the discrete time step δt. By expanding in Taylor series we can write

$$\begin{aligned}\delta\Pi &= V_t\,\delta t + V_S\,\delta S + \tfrac{1}{2}V_{SS}\,\delta S^2 - \Delta\,\delta S - (k_1 + k_2 N + k_3 NS) + \dots \\ &= V_t\,\delta t + (V_S - \Delta)\,\delta S + \tfrac{1}{2}\sigma^2 S^2\phi^2\,\delta t V_{SS} - (k_1 + k_2 N + k_3 NS) + \dots.\end{aligned}$$

Note the similarity to the usual Black–Scholes argument. There are important differences however. First, we have subtracted off the transaction costs; they cause a reduction in the value of the portfolio. Second, the V_{SS} term is multiplied by the square of the random variable ϕ; we cannot appeal to Itô's lemma in this discrete time world.

Now we come to choosing the hedging strategy. The Leland strategy is to take

$$\Delta = V_S, \tag{5}$$

as in Black–Scholes, and rehedge every time step. (The choice (5) minimizes the variance, and hence the risk, of the portfolio to the order of magnitude we are considering.)

We now have

$$\delta\Pi = V_t\,\delta t + \tfrac{1}{2}\sigma^2 S^2\phi^2\,\delta t V_{SS} - (k_1 + k_2 N + k_3 NS) + \dots.$$

The change in the portfolio value, $\delta\Pi$, still contains elements of risk, in the ϕ^2 term and the transaction cost term. The latter is because we do not know N. For this reason we work in terms of the expected change in the value of the portfolio. Thus

$$E[\delta\Pi] = V_t\,\delta t + \tfrac{1}{2}\sigma^2 S^2\,\delta t V_{SS} - (k_1 + k_2 E[N] + k_3 E[N]S) + \dots. \tag{6}$$

The only unknown term in (6) is $E[N]$, and this is easily calculated. Because $\Delta = V_S(S,t)$, we know that

$$N = |\Delta(S+\delta S, t+\delta t) - \Delta(S,t)|;$$

this is the change in the number of shares held. Thus

$$N = |V_S(S+\delta S, t+\delta t) - V_S(S,t)|,$$

and, to leading order,
$$N = |V_{SS}\,\sigma S\phi\,\delta t^{\frac{1}{2}}|.$$

From this it follows that

$$E[N] = \sqrt{(2/\pi)}\,\sigma S\,\delta t^{\frac{1}{2}}|V_{SS}|.$$

We now value the option by setting the return on the portfolio equal to that from a risk-free deposit. This is our valuation policy; we have eliminated risk to the best of our ability given our hedging strategy and we attach no further value to accepting the remaining risk. We thus find that

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = (k_1/\delta t) + (k_2 + k_3 S)\sqrt{(2/\pi\delta t)}\,\sigma S|V_{SS}|, \qquad (7)$$

where we have dropped terms of $O(\delta t^{\frac{1}{2}})$ or smaller.

(c) *Other models and matters arising*

The above model is an extended version of that due to Leland. The correction due to the effect of transaction costs depends on the second derivative of the option price with respect to the asset price. This derivative is commonly referred to as the 'gamma', and it is a measure of the mishedging due to the discreteness of the hedging, and hence a measure of the level of transaction costs.

Because of the significance of V_{SS}, many transaction cost models result in equations of the form

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = F(S, V_{SS}),$$

where the function $F(\cdot,\cdot)$ depends upon the hedging strategy. We shall now give some examples.

(i) Extended Leland (see Hoggard *et al.* 1993): rehedge to Black–Scholes delta every δt,

$$F(S,x) = (k_1/\delta t) + (k_2 + k_3 S)\sqrt{(2/\pi\delta t)}\,\sigma S|x|.$$

(ii) Market practice model (see Whalley & Wilmott 1992): continuous time, rehedge to Black–Scholes value when delta moves outside a 'hedging bandwidth' of size ϵ/S, where ϵ is a function of S, t as well as V and its derivatives,

$$F(S,x) = (\sigma^2 S^4/\epsilon)\,[k_1 + (k_2 + k_3 S)\,\epsilon^{\frac{1}{2}}/S]\,x^2.$$

A special case of this is $\epsilon = \sigma^2 S^4 V_{SS}^2\,\delta t^{\frac{1}{2}}$, in which case the expected time between rehedges is constant and the model reduces to the extended Leland.

(iii) Small costs limit of Davis *et al.* (1993) model: this model is in continuous time and is based upon utility maximization,

$$F(S,x) = \frac{e^{-r(T-t)}}{\gamma}\left(\frac{3k_3\gamma^2 S^4\sigma^3}{8\delta^2}\right)^{\frac{2}{3}}\left(\left|x - \frac{e^{-r(T-t)}(\mu - r)}{\gamma S^2\sigma^2}\right|\right)^{\frac{4}{3}}.$$

Here γ is the index of risk aversion and $\delta = e^{-r(T-t)}$. Note that this problem depends on the growth rate μ.

In Whalley & Wilmott (1992) many issues arising from such equations are discussed. Briefly, these include the following.

1. Nonlinearity. As the right-hand side of the equation is in each case a nonlinear function of the Black–Scholes value of gamma, V_{SS}, there will inevitably be different

values for short and long positions. Also portfolios of options should be treated as a whole and not as the sum of individually valued components. There will be advantage to be gained from offsetting opposite positions as well as economies of scale.

2. Negative option prices. With the general cost structure discussed in Hoggard *et al.* (1993) and Whalley & Wilmott (1992) (not simply bid-offer spread) it is possible to arrive at negative option prices. (To see this, consider the commission component of costs. If a fixed amount is paid at each rehedge then for small asset values it may cost more to hedge a call than the call is worth.) This suggests modifying hedging strategies to allow the possibility of not rehedging if to rehedge would make the option value negative. This introduces a free boundary below which (for a call) the option should not be rehedged.

3. American options. As also mentioned in Davis *et al.* (1993), it is the owner of the American option who controls its exercise. It is difficult to value an American option optimally unless the owner's hedging and exercise strategy is known. This entails at least knowing all of the owner's estimates of the parameters.

The strong nonlinearities associated with the transaction cost term $F(S, V_{SS})$ can lead to ill-posedness. We interpret this as meaning that the hedging strategy is inappropriate. As an example, we consider the market practice model (Whalley & Wilmott 1992) with $k_1 \ll 1$ and $k_2 = k_3 = 0$. Thus we have

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = (k_1 \sigma^2 S^4/\epsilon)(V_{SS})^2.$$

We assume that $\delta = k_1 \sigma^2/\epsilon \ll 1$ so that transaction costs appear as a small perturbation to the Black–Scholes equation. We see, however, that the sign of the curvature, that is, the sign of the gamma, is critically important. Solving for V_{SS} we find that

$$S^2 V_{SS} = \tfrac{1}{2}\sigma^2/\delta[1 \pm \surd(1 + (4\delta/\sigma^4)(V_t + rSV_S - rV))].$$

On the one hand, if $V_{SS} < 0$ then

$$S^2 V_{SS} \sim -(1/\sigma^2)(V_t + rSV_S - rV) + O(\delta),$$

and we have a backward parabolic problem with final conditions. This is well posed. Moreover, in this situation we are justified in looking for a regular asymptotic expansion,

$$V = V_0 + \delta V_1 + O(\delta^2),$$

which treats the hedging cost as a small perturbation to the Black–Scholes model. If the gamma is negative we can use this model to value the option.

On the other hand, if $V_{SS} > 0$ we have

$$S^2 V_{SS} \sim (\sigma^2.\delta) + (1/\sigma^2)(V_t + rSV_S - rV) + O(\delta),$$

which then gives us a forward parabolic problem with final conditions. This is ill posed. Moreover, we see that we must look for asymptotic solutions of the form

$$V = (V_{-1}/\delta) + V_0 + I(\delta),$$

whenever V_{SS} is $O(\delta^{-1})$, as in the case of an at the money call or put near expiry. Thus there is a blow up (associated with the ill-posedness) which will, in most cases, occur over a very short time scale in the neighbourhood of the expiry date. This shows that we cannot value an option hedged according to the market practice model if the gamma is large and positive.

4. Valuing exotic options in the presence of transaction costs

The modelling and analyses which led to the exotic option problem and the transaction cost problem separately may be formally combined to give the following nonlinear partial differential equation problem for exotic options in the presence of transaction costs.

$$V_t + fV_I + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (rS - D)\, V_S - rV = F(S, V_{SS}), \tag{8}$$

where we can take F to be any of the functions in the previous section depending on the hedging strategy.

(a) *Similarity solutions*

In general the solution for the value of the path-dependent options we have mentioned must be solved numerically and usually in the three dimensions of asset price, path-dependent quantity and time. In some special cases however there are similarity reductions that reduce the number of dimensions to only two. This significantly decreases the computing time. Whether there is a similarity reduction not only depends on the pay-off but also on the nature of the hedging strategy and the type of costs involved. This is easily demonstrated by example.

It is well known and easily seen that in the absence of transaction costs the lookback put having pay-off

$$\max(J - S, 0),$$

where J is the maximum, has a similarity solution of the form

$$V(S, J, t) = JH(S/J, t).$$

There is still a similarity solution of this form in the Leland hedging model with only the bid-offer spread cost. If either of the k_1 and k_2 cost terms are non-zero then this solution is rendered invalid.

Another example is the average strike foreign exchange option having pay-off

$$\max[1 - (I/TS), 0],$$

where

$$I = \int_0^t S\, \mathrm{d}\tau.$$

This has a similarity solution of the form

$$V(S, I, t) = H(S/I, t)$$

in the absence of transaction costs and the same form in the Leland model when $k_2 = 0$.

Despite these examples being faster to compute they require special restrictions on the cost structure. For this reason the examples we shall shortly be giving are all genuinely three-dimensional.

(b) *Discrete sampling and discrete dividends*

We have set up an exotic pricing model that allows for discretely paid dividends and discretely sampled path-dependent quantities. We have then introduced transaction costs and described the small effects of transaction costs. Are these consistent? Let us consider the case of discretely paid dividends; the following idea carries over directly to the discretely sampled exotic case.

Recall from earlier that across a dividend state the realized option price is continuous but V as a function of S jumps. This is because the realized path of the asset is itself discontinuous across a dividend date. Even though the realized option price is continuous, the delta of the option (its derivative with respect to the underlying asset) is discontinuous. Thus across a dividend date there must be a rehedge of the order of the dividend yield. Each time there is a dividend payment there is a rehedge of this order. Typically, therefore, the total cost associated with these relatively large rehedges is, in the Leland model for example, of order

$$Mk_1 + (k_2/S) \times \text{total dividend} + k_3 \times \text{total dividend},$$

where M is the number of dividend payments. On the other hand, we have already seen that the rehedging over each period δt leads to the total costs over the life of the option of order

$$(k_1 T/\delta t) + [k_2 T\sigma/\surd(\delta t)] + [k_3 ST\sigma/\surd(\delta t)].$$

From these it can be seen that in practice, the total costs associated with a small number of discrete dividend payments are small compared with the accumulation of costs due to the frequent rehedging between dividend dates. We shall thus ignore them.

5. Results

In this section we consider numerically computed solutions of the generalized Leland model for lookback strike puts. The maxima, J, for the underlying are assumed to be discretely sampled, that is

$$J = \max_i \{S(t_i)\},$$

where t_i are the sampling dates. The solutions have been calculated using explicit finite differences, which are necessary in view of the strong nonlinearities introduced by the transaction costs. The jump condition across a maximum sampling date is

$$V(S, J, t^-) = V(S, \max(J, S), t^+)$$

(see, for example, Dewynne & Wilmott 1991). Note that this implies we must solve the problem on a square grid in the (S, J) plane. An implication is that we cannot use asymptotic estimates for $V(S, J, t)$, which apply only for $S \gg J$, when S and J are comparable. We overcome this difficulty by using explicit finite differences and not applying boundary conditions for large S. Rather, at each time step we discarded the value of V for the largest values of S and J, that is, our square grid shrank by one mesh step at each time step.

In figures 1, 2 and 3 we value long positions in discretely sampled lookback strike puts. The pay-offs are

$$\max(J - S, 0).$$

The lifetimes of the options are all $T = 0.25$, the annual volatility is 0.2, the annual interest rate is 0.05 and there are eight discrete times at which the maximum is sampled. These are at $t = 0.225$, $t = 0.200$, and so on back to $t = 0.025$. In all cases we assume the option value may not become negative.

In figure 1 there are no transaction costs, $k_1 = k_2 = k_3 = 0$. We plot the option

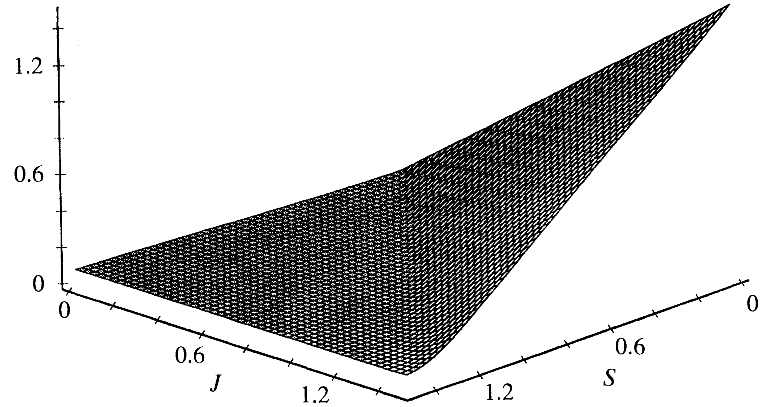

Figure 1. The value of a discretely sampled lookback strike put with pay-off max $(J-S,0)$ as a function of J and S.

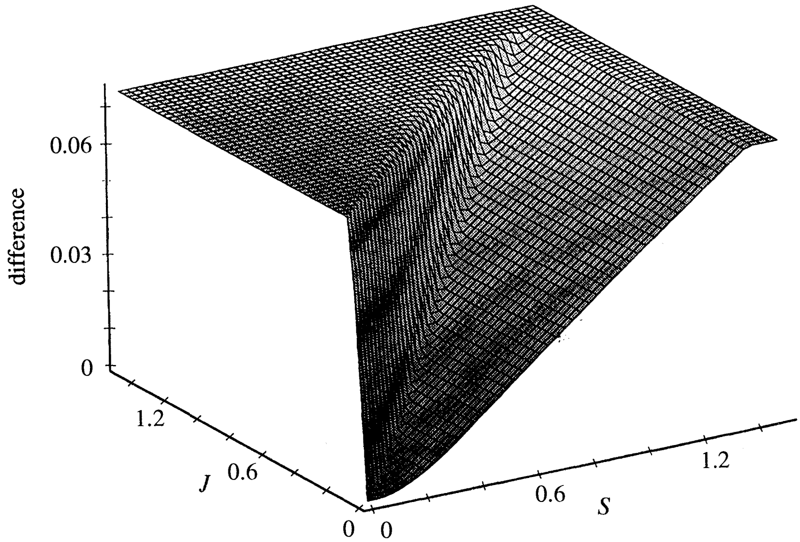

Figure 2. The difference between the Black–Scholes and with-cost value of a discretely sampled lookback strike put with payoff max $(J-S,0)$ as a function of J and S.

value against S and J. This problem has a similarity solution with independent variable S/J. If V/J is plotted against this variable then the picture would show a minimum close to $S/J=1$. The delta for this option thus changes sign. This is because the holder of the option benefits from a large value of the maximum J established before expiry but a small value of S at expiry, and, of course, these two are not independent.

In figure 2 there are fixed transaction costs only; $k_1/\delta t=0.3$, but $k_2=k_3=0$. We plot the difference between the Black–Scholes (zero cost) value and the solution with costs. The difference is always positive; the option's value has been decreased. The effect of the fixed transaction cost is to decrease the option value everywhere by the same amount. This gives negative option values for certain S and J. However, as soon as the positivity constraint is added the option value is no longer decreased everywhere by the same amount; it is decreased by less at some places. This explains the dip in the plot. This problem does not have a similarity solution. As discussed

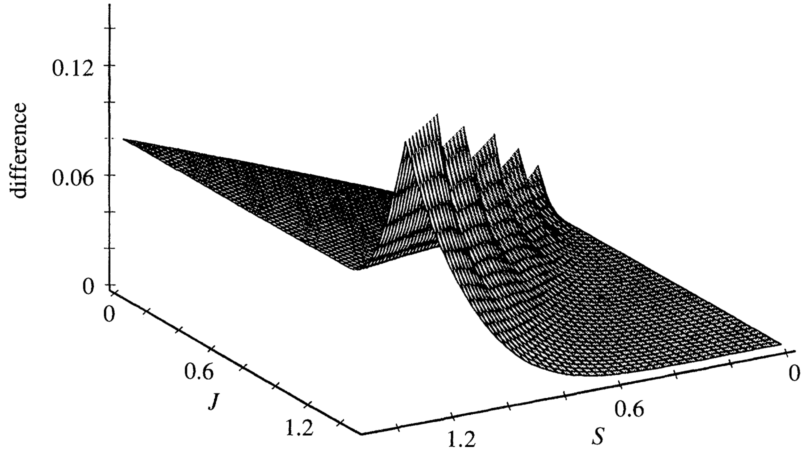

Figure 3. The difference between the Black–Scholes and with-cost value of a discretely sampled lookback strike put with payoff max $(J-S, 0)$ as a function of J and S.

above the option price would naturally become negative for some values of S and J. Here we have imposed the constraint $V \geqslant 0$. For some values of S and J the option price is zero; we have solved a free boundary problem.

In figure 3 there are only costs proportional to the volume of underlying traded, $k_1 = 0$, $k_2 \sigma \surd(2/\pi\delta t) = 0.3$ and $k_3 = 0$. Again we have plotted the difference between the theoretical Black–Scholes value and the value allowing for costs. The effect of costs is again to decrease the value of the option. For small S the cost effect tends to zero, since it is proportional to the option's gamma. For large S the option value is zero, and so the difference is simply the large S behaviour of the Black–Scholes solution. Again there is no similarity solution. Since the second derivative V_{SS} is positive the diffusion term vanishes at

$$\tfrac{1}{2}\sigma^2 S^2 = k_2 S \sigma \surd(2/\pi\delta t),$$

that is

$$S = (2k_2/\sigma) \surd(2/\pi\delta t).$$

We thank the Royal Society and Adam, Harding & Lueck for their support.

References

Atkinson, C. & Wilmott, P. 1993 Portfolio management with transaction costs: an asymptotic analysis. *Math. Finance.* (In the press.)

Black, F. & Scholes, M. 1973 The pricing of options and corporate liabilities. *J. Political Econ.* **81**, 637–654.

Boyle, P. P. & Vorst, T. 1992 Option replication in discrete time with transaction costs. *J. Finance* **47**, 271.

Davis, M. H. A., Panas, V. G. & Zariphopoulou, T. 1993 European option pricing with transaction costs. *SIAM Jl. Control* **31**, 470–493.

Dewynne, J. N. & Wilmott, P. 1991 Lookback options. OCIAM, Oxford (Working paper).

Dewynne, J. N. & Wilmott, P. 1993 Partial to the exotic. *Risk* March, 38–46.

Hodges, S. D. & Neuberger, A. 1993 Optimal replication of contingent claims under transaction costs. *Rev. Future Markets* **8**, 222–239.

Hoggard, T., Whalley, A. E. & Wilmott, P. 1993 Hedging option portfolios in the presence of transaction costs. *Adv. Fut. Opt. Res.* (In the press.)

Ingersoll, J. 1987 *Theory of financial decision making*. Rowman and Littlefield.

Leland, H. E. 1985 Option pricing and replication with transaction costs. *J. Finance* **40**, 1283–1301.

Morton, A. & Pliska, S. R. 1993 Optimal portfolio management with fixed transaction costs. *Math. Finance*. (In the press.)

Whalley, A. E. & Wilmott, P. 1992 A hedging strategy and option valuation model with transaction costs. OCIAM, Oxford (Working paper).

Wilmott, P., Dewynne, J. N. & Howison, S. D. 1993 *Option pricing: mathematical models and computation*. Oxford Financial Press.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

THE ROYAL SOCIETY A

PHILOSOPHICAL TRANSACTIONS OF